Fitted Finite Volume Method for a Generalized Black-Scholes Equation Transformed on Finite Interval

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Abstract. A generalized Black-Scholes equation is considered on the semi-axis. It is transformed on the interval $(0,1)$ in order to make the computational domain finite. The new parabolic operator degenerates at the both ends of the interval and we are forced to use the Gärding inequality rather than the classical coercivity. A fitted finite volume element space approximation is constructed. It is proved that the time θ-weighted full discretization is uniquely solvable and positivity-preserving. Numerical experiments, performed to illustrate the usefulness of the method, are presented.

Keywords: generalized Black-Scholes equation, degenerate parabolic equation, Gärding coercivity, fitted finite volume method, positivity

1 Introduction

The famous equation, proposed by F. Black, M. Scholes and R. Merton, see [13,19], is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\left(\frac{\partial V}{\partial S} - SV\right) = 0,\ S \in (0,\infty),\ t \in [0,T).$$

This is a typical example of a *degenerate* parabolic equation [14]. It is well known [13,19] that it can be transformed to the heat equation that allows us to overcome the degeneracy at $S = 0$. Many numerical methods, based on classical finite difference schemes, applied to constant coefficients heat equations, are developed [1,15]. However, when the problem has space-dependant coefficients σ and r one can not transform the Black-Scholes equation to a standard heat equation. Finite difference and finite element methods have been applied in [2,3,4,5,6,7,8,10,16,21] in order to solve this type of generalized Black-Scholes equations. In [11] cubic B-splines are implemented. Often, the convergence of the full discretization is verified by numerical examples only.

An effective method, that resolves *the degeneracy*, is proposed by S. Wang [20] for the Black-Scholes equation with Dirichlet boundary conditions. The method

is based on a finite volume formulation of the problem, coupled with a fitted local approximation to the solution and an implicit time-stepping technique. The local approximation is determined by a set of two-point boundary value problems (BVPs), defined on the element edges. This fitted technique originates from one-dimensional computational fluid dynamics [12].

A modification of the discretization, originally presented in [20], was proposed by L. Angermann [2] such that the method adequately treats the proper (natural) boundary condition at $x = 0$. Similar space discretization is derived in [6] for a degenerate parabolic equation in the zero-coupon bond pricing.

The domain of S is the half real line. For numerical computation it is desirable to have a *finite computational domain*. The transformation in the next section converts $S \in (0, \infty)$ to $x \in (0, 1)$, *decreasing significantly the computational costs*. Also, for a call option, the solution $V(S,t)$ is not bounded and from the numerical methods' point of view the problem transformation on a finite interval is better. The resulting equation has variable coefficients but this is not an essential difficulty for the numerical computation. However, the transformed equation *degenerates* at both ends of the finite interval.

The present paper deals with a degenerate parabolic equation, (4), derived after transformation of the generalized Black-Scholes equation (1) to a finite interval. The degeneration at the both ends of the interval does not allow the use the Poincaré-Friedrichs inequality and we are forced to investigate the differential problem with the Gärding inequality rather than classical coercivity [9].

This paper is organized as follows. The model problem is presented in Section 2, where we discuss our basic assumptions and some properties of the solution. The space discretization method is developed in Section 3. Section 4 is devoted to the time discretization, where we show that the system matrix on each time-level is an M-matrix so that the discretization is monotone. Numerical experiments are discussed in the last section.

Some results, concerning the case of the transformed Black-Scholes equation above, are reported in [17].

2 The transformed problem

We consider the generalized Black-Scholes equation [13,19]:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 V}{\partial S^2} + (r(t)S - D(S,t))\frac{\partial V}{\partial S} - r(t)V = 0,\ (S,t) \in (0,\infty)\times(0,T), \tag{1}$$

where $\sigma = \sigma(t)$ denotes *the volatility* of the asset, $r = r(t)$ is *the risk-free interest rate*, $D = D(S,t)$ denotes *the dividend* of the dividend-paying asset. We also introduce $d = d(S,t)$ such that $D(S,t) = Sd(S,t)$, where *the dividend rate* $d = d(s,t)$ is continuously differentiable with respect to S.

There are various choices for the final (payoff) condition, depending on the models. In the case of vanilla European option

$$V(S,T) = \begin{cases} \max(S-E), \text{ for a call option,} \\ \max(E-S), \text{ for a put option,} \end{cases} \tag{2}$$

where E is the strike price. Another example is the *bullish vertical spread* payoff, defined by

$$V(S,T) = max(S-E_1) - max(S-E_2), \tag{3}$$

where E_1 and E_2 are two exercise prices, satisfying $0 < E_1 < E_2$. This represents a portfolio of buying one call option with exercise price E_1 and issuing one call option with the same expiry date but a larger exercise price E_2. For detailed discussion on this, we refer to [19].

We introduce the transformation [21]

$$x = \frac{S}{S+P_m},\ u(x,t) = \frac{V(S,t)}{S+P_m},\ \tau = T - t.$$

The constant P_m is called *mesh parameter*. It controls the distribution of the mesh nodes w.r.t. S on the interval $(0,\infty)$. The higher the value of S, that we are interested in, the higher value of P_m should be in order to obtain a reasonable accuracy. In the case of a call option, because of the nature of the terminal condition, P_m should be equal to E.

The inverse transformation is $S = P_m x/(1-x)$ and after plugging it in the Black-Scholes equation, (1) we obtain:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2(t)x^2(1-x)^2\frac{\partial^2 u}{\partial x^2} - x(1-x)(r(t)-d(x,t))\frac{\partial u}{\partial x} \\ &+((1-x)r(t)+xd(x,t))u = 0,\ (x,t)\in \Omega_T = \Omega\times(0,T), \Omega = (0,1). \end{aligned} \tag{4}$$

We return to the original notation of the variable t for the sake of simplicity. The initial data for a call option reads

$$u(x,0) = u_0(x) = \max(2x-1,0). \tag{5}$$

Being different from the classical parabolic equations, in which the principle coefficient is assumed to be strictly positive, the parabolic equation (4) belongs to the second order differential equations with non-negative characteristic form [14]. The main difficulty of such kind of equations is *the degeneracy* [18]. It can be easily seen that at $x = 0$ and $x = 1$ (4) degenerates to

$$\left.\frac{\partial u}{\partial t}\right|_{x=0} = -r(t)u(0,t),\ \left.\frac{\partial u}{\partial t}\right|_{x=1} = -d(1,t)u(1,t).$$

It is well known by the Fichera theory for degenerate parabolic equations [14] that at the degenerate boundaries $x = 0$ and $x = 1$ the boundary conditions should not be given.

For theoretical analysis of our discrete problem as well as for the construction of a fitted finite volume mass-lumping discretization we need to consider weak solutions of (4). We shall use the standard notations for the function space

$C^m(\Omega)$ and $C^m(\overline{\Omega})$ of which a function and it's derivatives up to order m are continuous on Ω (respectively $\overline{\Omega}$). The space of square-integrable functions we denote by $L^2(\Omega)$ with the usual L^2-norm $\|\cdot\|$ and the inner product $(\cdot,\cdot)$. We also use the function space $L^\infty(\Omega)$ with the norm $\|\cdot\|_\infty$. To handle the degeneracy in (4), we introduce the following weighted L^2_w-norm

$$\|v\|_{0,w} := (\int_0^1 x^2(1-x)^2 v^2 dx)^{1/2}$$

with corresponding inner product $(u,v)_w$. Using $L^2(\Omega)$ and $L^2_w(\Omega)$, we define the weighted Sobolev space $H^1_w(\Omega) := \{v : v \in L^2(\Omega),\ v' \in L^2_w(\Omega)\}$, where v' denotes the weak derivative of v. Let $\|\cdot\|_{1,w}$ be the functional on $H^1_w(\Omega)$, defined by $\|v\|_{1,w} = (\|v\|_0^2 + \|v'\|^2_{0,w})^{1/2}$. Then it is easy to see that $\|\cdot\|_{1,w}$ is a norm on $H^1(\Omega)$; it is called weighted H^1-norm on $H^1_w(\Omega)$. Furthermore, using the inner products in $L^2(\Omega)$ and $L^2_w(\Omega)$, we define a weighted inner product on $H^1_w(\Omega)$ by $(\cdot,\cdot)_H := (\cdot,\cdot) + (\cdot,\cdot)_w$ and, consequently, the pair $(H^1_w(\Omega), (\cdot,\cdot)_H)$ is a Hilbert space. Also, $H^1_w(\Omega)$ contains the conventional Sobolev space $H^1(\Omega)$ as a proper subspace.

We rewrite (4) in divergent form

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(x(1-x)\left(a(x,t)\frac{\partial u}{\partial x} + b(x,t)u\right)\right) + c(x,t)u = 0,\\
&a(x,t) = \frac{1}{2}\sigma^2(t)x(1-x),\ b(x,t) = \left(r(t) - d(x,t) + \sigma^2(t)(2x-1)\right), \qquad (6)\\
&c(x,t) = (2-3x)r(t) - (6x^2-6x+1)\sigma^2(t) - (1-3x)d(x,t) - x(1-x)\frac{\partial d}{\partial x}.
\end{aligned}$$

Let us introduce for $w, v \in H^1_w(\Omega)$ the bilinear form

$$A(w,v;t) := (aw' + bw, v') + (cw, v) = (x(1-x)\rho(w), v') + (cw, v).$$

Here the notation $w' = \frac{\partial w}{\partial x}$ is used and the function $\rho(w) = aw' + bw$ is *the weighted flux density* associated with w. We are in position to state the variational formulation of (4):

Find $u(t) \in H^1_w(\Omega)$, such that for all $v \in H^1_w(\Omega)$

$$\left(\frac{\partial u(t)}{\partial t}, v\right) + A(u(t), v; t) = 0 \text{ a.e. in } (0,T) \text{ and } (\cdot,0) = u_0. \qquad (7)$$

The following result provides *the weak coercivity and continuity* of the bilinear form $A(\cdot,\cdot,t)$.

Lemma 1. *Let $w, v \in C^1([0,1])$. Then:*

1. *there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$|A(w,v,t)| \le C_1 \|w\|_{H^1_w(0,1)} \|v\|_{H^1(0,1)} \qquad (8)$$
$$|A(w,v,t)| \le C_2 \|w\|_{H^1(0,1)} \|v\|_{H^1_w(0,1)} \qquad (9)$$

for any $t \in [0,T]$;

2. ***(Gärding inequality)*** *there exist constants* $\alpha > 0$ *and* $\gamma > 0$ *such that*

$$|A(v, v, t)| \geq \alpha \|v\|^2_{H^1_w(0,1)} - \gamma \|v\|^2_{L^2(0,1)}, \tag{10}$$

uniformly with respect to $t \in [0, T]$.

Proof. The proof is given in [9].

Owing to Lemma 1 one can prove the following assertion [9].

Theorem 1. *Suppose that* $u_0(x) \in H^1_w(\Omega)$. *Then the problem* (7) *has an unique solution.*

Theorem 2. *(**Weak maximum principle**) Let* $u \in L^2\left(0, T; H^1_w(\Omega)\right)$ *be such that*

1. $\frac{\partial u}{\partial t} \in L^2\left((0, 1) \times (0, T)\right)$;
2. *the inequality*

$$\left(\frac{\partial u(t)}{\partial t}, v\right) + A(u(t), v; t) \geq 0, \qquad \forall v \in C^\infty_0\left(0,1\right), v \geq 0$$

holds for a.e. $\tau \in [0, T]$;

3. $u|_{t=0} \geq 0$.

Then $u \geq 0$ *a.e. in* $(0, 1) \times (0, T)$.

Proof. The proof is given in [9].

3 Space discretization

In this section we describe the finite volume approximation of (4).

Let the interval $\Omega = (0, 1)$ be subdivided into N intervals $I_i := (x_i, x_{i+1}),\ i = 0, \ldots, N-1$, with $0 =: x_0 < x_1 < \cdots < x_N := 1$. For each $i = 0, \ldots, N-1$ we set $h_i := x_{i+1} - x_i$ and $h := \max_{i=0,\ldots,N-1} h_i$. We also denote $x_{i+1/2} := (x_i + h_i/2)$ for $i = 0, \ldots, N-1$, $x_{-1/2} := x_0 = 0$, $x_{N+1/2} := x_N = 1$ and $\Omega_i := (x_{i-1/2}, x_{i+1/2})$ for $i = 0, \ldots, N$. Finally, we define $l_i := x_{i+1/2} - x_{i-1/2}$ for $i = 0, \ldots, N$.

Integrating (7) over the control volumes Ω_i we obtain $N+1$ balance equations

$$\int_{\Omega_i} \dot{u} dx - [x(1-x)\rho(u)]^{x_{i+1/2}}_{x_{i-1/2}} + \int_{\Omega_i} cu dx = 0, \ \ i = 0, \ldots, N.$$

Multiplying the i-th equation with an arbitrary number v_i and summing up the results we get

$$\sum_{i=0}^{N} \int_{\Omega_i} \dot{u} v_i dx - \sum_{i=0}^{N} [x(1-x)\rho(u)]^{x_{i+1/2}}_{x_{i-1/2}} v_i + \sum_{i=0}^{N} \int_{\Omega_i} cuv_i dx = 0. \tag{11}$$

For an arbitrary function $v \in C(\overline{\Omega})$ we define the mass-lumping operator $L : C(\overline{\Omega}) \to L_\infty(\Omega)$ by $L_h v|_{\Omega_i} := v(x_i),\ i = 0, \ldots, N$.

Therefore, using the operator L_h, equation (11) can be written as follows:

$$(\dot{u}(t), L_h v) + A_h(u, v; t) = 0,\ \forall v \in C(\overline{\Omega}),$$
$$A_h(w, v; t) := -\sum_{i=0}^{N} \left[x(1-x)\rho(w, x, t)\right]_{x_{i-1/2}}^{x_{i+1/2}} L_h v(x_i) + (c(t)w, L_h v).$$

The spatial discretization starts from this equation. Applying the mid-point quadrature rule to all terms except the second one we obtain for all $v \in C(\overline{\Omega})$

$$(\dot{u}(t), v)_h - \sum_{i=0}^{N} \left[x(1-x)\rho(u(t), x, t)\right]_{x_{i-1/2}}^{x_{i+1/2}} L_h v(x_i) + (c(t)u, v)_h = 0,$$
$$(w, v)_h := (L_h w, L_h v) = \sum_{i=0}^{N} w_i v_i l_i,\ w, v \in C(\overline{\Omega}).$$

Clearly, we now need to derive approximations of the continuous weighted flux density $x(1-x)\rho(u(t), x, t)$, defined above, at the midpoints $x_{i+1/2}$ of the intervals I_i for $i = 0, \ldots, N-1$.

Case 1 Approximation of ρ at $x_{i+1/2}$ for $1 \le i \le N-2$.

Let us consider the following two-point boundary value problem for $x \in I_i$

$$(a_{i+1/2} x(1-x)v' + b_{i+1/2} v)' = 0;\ \ v(x_i) = u_i, v(x_{i+1}) = u_{i+1},$$

where $a_{i+1/2} = a(x_{i+1/2}),\ \ b_{i+1/2} = b(x_{i+1/2}, t)$.

Following considerations, similar to those in [6,17], we obtain

$$\rho_i(u) = b_{i+1/2} \frac{\left(\frac{x_{i+1}}{1-x_{i+1}}\right)^{\alpha_i} u_{i+1} - \left(\frac{x_i}{1-x_i}\right)^{\alpha_i} u_i}{\left(\frac{x_{i+1}}{1-x_{i+1}}\right)^{\alpha_i} - \left(\frac{x_i}{1-x_i}\right)^{\alpha_i}},\ \ i = 1, \ldots, N-2, \tag{12}$$

where $\alpha_i = \frac{b_{i+1/2}}{a_{i+1/2}}$ and $\rho_i(u)$ provides an approximation to $\rho(u)$ at $x_{i+\frac{1}{2}}$.

Case 2 Approximation of ρ at $x_{1/2}$.

Now we write the flux in the form

$$\rho(u) := \overline{a} x \frac{\partial u}{\partial x} + bu,\ \ \overline{a} = \overline{a}(x) = \frac{\sigma^2}{2}(1-x).$$

Note that the analysis in Case 1 can not be applied here because the flux degenerates near $x = 0$. To solve this difficulty, following [2,6,17,20], we will reconsider the flux ODE with an extra degree of freedom in the following form

$$(\overline{a}_{1/2} x v' + \overline{b}_{1/2} v)' = C \ \text{ in } \ (0, x_1),\ \ v(0) = u_0,\ \ v(x_1) = u_1,$$

where C is an unknown constant to be determined. We obtain

$$v(x) = \begin{cases} x\frac{u_1 - u_0}{h_0} + u_0, & \alpha_0 \ge 0, \\ \left(\frac{x}{h_0}^{(1+\alpha_0)}\right) x\frac{u_1-u_0}{h_0} + u_0, & \alpha_0 < 0, \end{cases}$$

$$\rho_0(v) := \begin{cases} \overline{a}(1+\alpha_0)x\frac{u_1-u_0}{h_0} + b_{1/2}u_0, & \alpha_0 \geq 0, \\ b_{1/2}u_0, & \alpha_0 < 0. \end{cases} \tag{13}$$

Case 3 Approximation of ρ at $x_{N-1/2}$.
We write the flux in the form

$$\rho(u) := \overline{\overline{a}}_{N-1/2}\,(1-x)\frac{\partial u}{\partial x} + b_{N-1/2}u,\ \overline{\overline{a}}\,(x) = \frac{\sigma^2}{2}x.$$

The situation is symmetric to Case 2. We can not apply the arguments in Case 1 to the approximation of the weighted flux density on $I_{N-1} = (x_{N-1}, x_N)$ because equation (4) degenerates at $x = x_N = 1$. However the considerations, given in Case 2, should be modified in order to formulate an appropriate two-point BVP. Again, we consider the flux ODE with an extra degree of freedom in the following form (recall $\alpha_{N-1} = b_{N-1/2}/\overline{\overline{a}}_{N-1/2}$)

$$((1-x)v' + \alpha_{N-1}v)' = C_0,\ x \in I_{N-1}, \tag{14}$$

$$v(x_{N-1}) = u_{N-1},\ v(x_N) = u_N, \tag{15}$$

where C_0 is an unknown constant to be determined. Integration of (14) yields the first-order linear equation

$$(1-x)v' + \alpha_{N-1}v = C_0x + C_1,\ x \in I_{N-1}, \tag{16}$$

where C_1 denotes an additive constant. We can easily obtain $\alpha_{N-1}u_N = C_0 + C_1$ and this would be legal if we solve the problem with Dirichlet boundary condition at $x = 1$. However, this is not the case and alternatively we multiply (16) by $(1-x)^{-\alpha_{N-1}-1}$

$$\left((1-x)^{-\alpha_{N-1}}v\right)' = C_0(1-x)^{-\alpha_{N-1}-1}x + C_1(1-x)^{-\alpha_{N-1}-1}. \tag{17}$$

Case 3.1 $\alpha_{N-1} > 0, \alpha_{N-1} \neq 1$.
Integrating (17) from x_{N-1} to $x \in I_{N-1}$ results in

$$\begin{aligned} &(1-x)^{-\alpha_{N-1}}v(x) - (1-x_{N-1})^{-\alpha_{N-1}}v(x_{N-1}) \\ &= C_0\ \frac{(1-s)^{-\alpha_{N-1}+1}}{-\alpha_{N-1}+1}\Big|_{x_{N-1}}^{x} - (C_0 + C_1)\ \frac{(1-s)^{-\alpha_{N-1}}}{-\alpha_{N-1}}\Big|_{x_{N-1}}^{x}. \end{aligned}$$

Multiplying both sides of the equation by $(1-x)^{\alpha_{N-1}}$

$$\begin{aligned} v(x) = {} & \tfrac{(1-x)^{\alpha_{N-1}}}{(1-x_{N-1})^{\alpha_{N-1}}}v(x_{N-1}) + C_0\tfrac{1-x}{-\alpha_{N-1}+1} - C_0\tfrac{(1-x_{N-1})^{-\alpha_{N-1}+1}(1-x)^{\alpha_{N-1}}}{-\alpha_{N-1}+1} \\ & -(C_0 + C_1)\tfrac{1}{-\alpha_{N-1}} + (C_0 + C_1)\tfrac{(1-x_{N-1})^{-\alpha_{N-1}}(1-x)^{\alpha_{N-1}}}{-\alpha_{N-1}}. \end{aligned}$$

Letting $x \to x_N = 1$ we arrive at $v(x_N) = u_N = \frac{C_0 + C_1}{\alpha_{N-1}}$ and finally

$$
\begin{aligned}
v(x) &= \tfrac{(1-x)^{\alpha_{N-1}}}{(1-x_{N-1})^{\alpha_{N-1}}}(u_{N-1}-u_N)+u_N \\
&+\tfrac{\omega}{-\alpha_{N-1}+1}(1-x)\left(1-\tfrac{(1-x)^{\alpha_{N-1}-1}}{(1-x_{N-1})^{\alpha_{N-1}-1}}\right),
\end{aligned}
$$

where $\omega = C_0 \in R$ is a free parameter. Therefore

$$
\rho_{N-1}(v) := \overline{\overline{a}}_{N-1/2}(x-1)\omega + b_{N-1/2}u_N.
$$

Case 3.2 $\alpha_{N-1} = 1$.

Now we solve the following ODE

$$
\left(\frac{v}{1-x}\right)' = C_0(1-x)^{-2}x + C_1(1-x)^{-2}.
$$

Integrating over $(x_{N-1}, x), x \in I_{N-1}$, we obtain

$$
\begin{aligned}
v(x) &= \tfrac{1-x}{1-x_{N-1}}v(x_{N-1}) + C_0(1-x)(\ln(1-x)-\ln(1-x_{N-1})) \\
&+(C_0+C_1)\left(\tfrac{1}{1-x}-\tfrac{1}{1-x_{N-1}}\right)(1-x).
\end{aligned}
$$

Letting $x \to x_N = 1$ one gets

$$
v(x_N) = u_N = C_0 + C_1 = \frac{C_0+C_1}{\alpha_{N-1}},
$$

$$
v(x) = \frac{(1-x)}{(1-x_{N-1})}(u_{N-1}-u_N) + u_N + \omega(1-x)\ln\frac{1-x}{1-x_{N-1}}.
$$

Since $1-x>0$ we can conclude that this is the result of the limiting process $\alpha_{N-1} \to 1$, performed on (18). The flux in both cases 3.1 and 3.2 can be written in the form

$$
\rho_{N-1}(v) = \overline{\overline{a}}_{N-1/2}(x-1)\omega + b_{N-1/2}u_N.
$$

Case 3.3 $\alpha_{N-1} = 0$.

Integrating over $(x_{N-1}, x), x \in I_{N-1}$ the following ODE

$$
v' = C_0(1-x)^{-1}x + C_1(1-x)^{-1}
$$

we arrive at

$$
v(x) = v(x_{N-1}) - C_0(x-x_{N-1}) - (C_0+C_1)(\ln(1-x)-\ln(1-x_{N-1})).
$$

The function $v(x)$ is bounded for $x \to x_N$ when $C_0 + C_1 = 0$. Therefore $C_0 = \frac{u_{N-1}-u_N}{1-x_{N-1}}$ and

$$v(x) = u_N + (u_{N-1} - u_N)\left(\frac{1-x}{1-x_{N-1}}\right).$$

The flux has the following form

$$\rho_{N-1}(v) = \overline{\overline{a}}_{N-1/2}(1-\alpha_{N-1})(1-x)\frac{u_N - u_{N-1}}{h_{N-1}} + b_{N-1/2}u_N,$$

where we used that $\alpha_{N-1} = b_{N-1} = 0$.

Case 3.4 $\alpha_{N-1} < 0$.

This time we integrate from x to $x_N = 1$ and obtain

$$v(x) = C_0\frac{(1-x)}{1-\alpha_{N-1}} + (C_0 + C_1)\frac{1}{\alpha_{N-1}}$$

and using the boundary conditions

$$C_0 = \frac{u_{N-1} - u_N}{1 - x_{N-1}}(1-\alpha_{N-1}),\ C_1 = \alpha_{N-1}u_N - \frac{u_{N-1} - u_N}{1-x_{N-1}}(1-\alpha_{N-1}).$$

Therefore

$$v(x) = \left(\frac{1-x}{1-x_{N+1}}\right)(u_{N-1} - u_N) + u_N,$$

$$\rho_{N-1}(v) = \overline{\overline{a}}_{N-1/2}(1-\alpha_{N-1})(1-x)\frac{u_N - u_{N-1}}{h_{N-1}} + b_{N-1/2}u_N$$

and these are exactly the same results as in Case 3.3. Finally, a reasonable choice of the free parameter ω is 0 and

$$v(x) = \begin{cases} u_N + \frac{1-x}{1-x_{N-1}}(u_{N-1} - u_N), & \alpha_{N-1} \le 0, \\ u_N + \frac{(1-x)^{\alpha_{N-1}}}{(1-x_{N-1})^{\alpha_{N-1}}}(u_{N-1} - u_N), & \alpha_{N-1} > 0, \end{cases}$$

$$\rho_{N-1}(v) = \begin{cases} \overline{\overline{a}}_{N-1/2}(1-\alpha_{N-1})(1-x)\frac{u_N - u_{N-1}}{h_{N-1}} + b_{N-1/2}u_N, & \alpha_{N-1} \le 0, \\ b_{N-1/2}u_N, & \alpha_{N-1} > 0. \end{cases}$$

Let us introduce the finite element space V_h by specifying it's basis $\{\phi_i\}_{i=0}^N$. Following [6,17] we introduce the functions

$$\phi_i(x) = \begin{cases} \dfrac{\left(\frac{1}{x_{i-1}}-1\right)^{\alpha_{i-1}} - \left(\frac{1}{x}-1\right)^{\alpha_{i-1}}}{\left(\frac{1}{x_{i-1}}-1\right)^{\alpha_{i-1}} - \left(\frac{1}{x_i}-1\right)^{\alpha_{i-1}}}, & x \in (x_{i-1}, x_i), \\ \dfrac{\left(\frac{1}{x_{i+1}}-1\right)^{\alpha_i} - \left(\frac{1}{x}-1\right)^{\alpha_i}}{\left(\frac{1}{x_{i+1}}-1\right)^{\alpha_i} - \left(\frac{1}{x_i}-1\right)^{\alpha_i}}, & x \in (x_i, x_{i+1}). \end{cases}$$

On the intervals $(0, x_1)$ and $(x_{N-1}, 1)$ we define the linear functions

$$\phi_0(x) = \begin{cases} 1 - \frac{x}{x_1}, \ x \in (0, x_1) \\ 0, \ \text{otherwise}; \end{cases} \quad , \quad \phi_N(x) = \begin{cases} \frac{(x - x_{N-1})}{(1 - x_{N-1})}, \ x \in (x_{N-1}, 1), \\ 0, \ \text{otherwise}. \end{cases}$$

Next we define the linear functions $\phi_1(x)$ and $\phi_{N-1}(x)$ on the intervals $(0, x_2)$ and $(x_{N-2}, 1)$

$$\phi_1(x) = \begin{cases} 1 - \frac{x}{x_1}, \ x \in (0, x_1), \\ \frac{\left(\frac{1}{x_2} - 1\right)^{\alpha_1} - \left(\frac{1}{x} - 1\right)^{\alpha_1}}{\left(\frac{1}{x_2} - 1\right)^{\alpha_1} - \left(\frac{1}{x_1} - 1\right)^{\alpha_1}}, \ x \in (x_1, x_2); \ \ 0, \ \text{otherwise}; \end{cases}$$

$$\phi_{N-1}(x) = \begin{cases} \frac{\left(\frac{1}{x_{N-2}} - 1\right)^{\alpha_{N-2}} - \left(\frac{1}{x} - 1\right)^{\alpha_{N-2}}}{\left(\frac{1}{x_{N-2}} - 1\right)^{\alpha_{N-2}} - \left(\frac{1}{x_{N-1}} - 1\right)^{\alpha_{N-2}}}, \ x \in (x_{N-2}, x_{N-1}), \\ (1 - x)/(1 - x_{N-1}), \ x \in (x_{N-1}, 1); \ \ 0, \ \text{otherwise}. \end{cases}$$

Then, for any $v_h \in V_h$, we have the representation $v_h(x) = \sum_{i=0}^{N} v_{hi}\phi_i$, where $v_{hi} := v_h(x_i)$. Associated with v_h, we introduce the natural interpolation operator $I_h : C(\overline{\Omega}) \to V_h$ by

$$I_h v_h(x_i) := v_h(x_i) = v_{hi}, \ \ i = 0, \ldots, N.$$

Furthermore, using the flux approximations, obtained in Cases 1, 2 and 3 respectively, we define by $\rho_h(u)$ an approximation to $\rho(u)|_{I_i} := \rho_i(u), \quad i = 0, \ldots, N-1$. Coming back to (7), this motivates the following semi-discretization of (6) in the space V_h:

$$(\dot{u}_h(t), v_h)_h + A_h(u_h(t), v_h; t) = 0 \ \ \forall v_h \in V_h,$$

where, using the long notation, i.e. $\rho_h(v_h, x, t) = \rho_h(v_h)$ with $\rho_h(v_h, x_i) = \rho_i(v_h)$ for $x \in I_i$

$$A_h(w_h, v_h; t) := -\sum_{i=0}^{N} [x(1-x)\rho_h(w_h, x, t)]_{x_{i-1/2}}^{x_{i+1/2}} L_h v_h(x_i) + (c(t)w_h, v_h)_h.$$

As usual, from (7) an equivalent ODEs system is obtained by setting successfully $v_h = \phi_i, \ i = 0, \ldots, N$:

$$\dot{u}_{hi}(t)l_i - x_{i+1/2}(1 - x_{i+1/2})\rho_h(u_h(t), x_{i+1/2}, t) + x_{i-1/2}\rho_h(u_h(t), x_{i-1/2}, t)$$
$$+ c_i(t)u_{hi}(t)l_i = 0, \ \ c_i(t) := c(x_i, t).$$

The complete set of equations forms an $(N+1) \times (N+1)$ system of linear ODEs w.r.t. $u_h(t) := (u_{h0}(t), \ldots, u_{hN}(t))^T$:

$$-M_h \dot{u}_h(t) + A_h(t)u_h(t) = 0,$$

where

$$M_h := ((\phi_j, \phi_i)_h)_{i,j=0}^{N} = diag(l_0, \ldots, l_N),$$
$$A_h(t) := (A_h(\phi_j, \phi_i; t))_{i,j=0}^{N} = (a_{ij}(t))_{i,j=0}^{N}.$$

4 Full discretization

Let $0 =: t_0 < t_1 < \dots t_{N_t} := T$ be a subdivision of the time interval $[0, T]$ with the step sizes $\triangle t_m := t_{m+1} - t_m > 0,\ m \in \{0, \dots, N_t - 1\}$. The fully discrete method with parameter $\theta \in [0, 1]$ for (6) reads as follows:

Find a sequence $U^1, \dots, U^{N_t} \in V_h$ such that for $m \in \{0, \dots, N_t - 1\}$

$$\begin{aligned}\left(\tfrac{U^{m+1}-U^m}{\triangle t_m}, v_h\right) &+ A_h(\theta U^{m+1} + (1-\theta)U^m, v_h; t_{m+\theta}) = 0\ \forall v_h \in V_h,\\ U^0 &= u_{0h},\end{aligned}$$

where $t_{m+\theta} := \theta t_{m+1} + (1-\theta)t_m = t_m + \theta \triangle t_m$ and $u_{0h} \in V_h$ is an approximation to u_0. By representing the elements U^m in terms of the basis $\{\phi_i\}_{i=0}^{N-1}$ of V_h and choosing $v_h = \phi_j,\ j = 0, \dots, N$ we get the algebraic form

$$\left(\frac{\mathbf{M}_h \mathbf{u}_h^{m+1} - \mathbf{M}_h \mathbf{u}_h^m}{\triangle t_m}\right) + \theta \mathbf{A}_h^{m+\theta} \mathbf{u}_h^{m+1} + (1-\theta)\mathbf{A}_h^{m+\theta}\mathbf{u}_h^m = 0,\ \mathbf{A}_h^m := \mathbf{A}_h(t_m). \tag{18}$$

The initial condition $\mathbf{u}_h^0$ is obtained from the representation of u_{0h} by means of the basis of V_h.

We will show, Theorem 3, that the system matrix $\mathbf{E}_h = \{e_{i,j}\}_{i,j=0}^{N,N_t}$ can be reduced to an M-matrix by excluding the first two and the last two equations in (18). Therefore, the above problem (18) is uniquely solvable and our method *preserves the positivity*, Theorem 2 (maximum principle), of the numerical solution in time. Let us introduce the notations

$$\begin{aligned}\varphi_i^{\alpha_i} &:= \left(\frac{x_i}{1-x_i}\right)^{\alpha_i},\ \varDelta_i^{\alpha_i} := \frac{1}{\varphi_{i+1}^{\alpha_i} - \varphi_i^{\alpha_i}},\\ a_{i\pm1/2} &:= a(x_{i\pm1/2}),\ b_{i\pm1/2}^{m+\theta} := b(x_{i\pm1/2}, t^{m+\theta}),\ c_{i\pm1/2}^{m+\theta} := c(x_{i\pm1/2}, t^{m+\theta})\end{aligned}$$

and write down the elements of the system matrix:

for $i = 0$ if $\alpha_0 < 0$ then

$$\begin{aligned}e_{0,0} &= \frac{l_0}{\varDelta t_m} + \theta x_{1/2}(1-x_{1/2})(-b_{1/2}^{m+\theta}) + \theta l_0 c_0^{m+\theta},\ e_{0,1} = 0\\ F_0 &= u_{h0}^m\left(\frac{l_0}{\varDelta t_m} - (1-\theta)\left(x_{1/2}(1-x_{1/2})(-b_{1/2}^{m+(1-\theta)}) + l_0 c_0^{m+(1-\theta)}\right)\right),\end{aligned}$$

and if $\alpha_0 \geq 0$ then

$$\begin{aligned}e_{0,0} &= \frac{l_0}{\varDelta t_m} + \theta x_{1/2}(1-x_{1/2})0.5(\overline{a}_{1/2} - b_{1/2}^{m+\theta}) + \theta l_0 c_0^{m+\theta},\\ e_{0,1} &= -\theta x_{1/2}(1-x_{1/2})0.5(\overline{a}_{1/2} + b_{1/2}^{m+\theta}),\\ F_0 &= u_{h0}^m\left(\frac{l_0}{\varDelta t_m} - (1-\theta)\left(x_{1/2}(1-x_{1/2})0.5(\overline{a}_{1/2} - b_{1/2}^{m+(1-\theta)}) + l_0 c_0^{m+(1-\theta)}\right)\right)\\ &\quad + u_{h1}^m(1-\theta)\left(x_{1/2}(1-x_{1/2})0.5(\overline{a}_{1/2} + b_{1/2}^{m+(1-\theta)})\right);\end{aligned}$$

for $i = 1$ if $\alpha_0 < 0$ then

$$
\begin{aligned}
e_{1,1} &= \frac{l_1}{\Delta t_m} + \theta x_{i+1/2}(1 - x_{i+1/2}) b_{i+1/2}^{m+\theta} \phi_i^{\alpha_i} \Delta_i^{\alpha_i} + \theta l_1 c_1^{m+\theta}, \\
e_{1,0} &= -\theta x_{1/2}(1 - x_{1/2})(-b_{1/2}^{m+\theta}), \\
e_{1,2} &= -\theta x_{i+1/2}(1 - x_{i+1/2}) b_{i+1/2}^{m+\theta} \phi_{i+1}^{\alpha_i} \Delta_i^{\alpha_i}, \\
F_1 &= u_{h1}^m \left(\frac{l_1}{\Delta t_m} - (1-\theta) \left(x_{i+1/2}(1 - x_{i+1/2}) b_{i+1/2}^{m+(1-\theta)} \phi_i^{\alpha_i} \Delta_i^{\alpha_i} + l_1 c_1^{m+(1-\theta)} \right) \right) \\
&+ u_{h0}^m (1-\theta) x_{1/2}(1 - x_{1/2})(-b_{i+1/2}^{m+(1-\theta)}) \\
&+ u_{h2}^m (1-\theta) x_{i+1/2}(1 - x_{i+1/2}) b_{i+1/2}^{m+(1-\theta)} \phi_{i+1}^{\alpha_i} \Delta_i^{\alpha_i}, \ i = 1 \text{ if } \alpha_0 < 0;
\end{aligned}
$$

and if $\alpha_0 \geq 0$ then

$$
\begin{aligned}
e_{1,1} &= \frac{l_1}{\Delta t_m} + \theta x_{i+1/2}(1 - x_{i+1/2}) b_{i+1/2}^{m+\theta} \phi_i^{\alpha_i} \Delta_i^{\alpha_i} \\
&+ \theta x_{1/2}(1 - x_{1/2}) 0.5(\overline{a}_{1/2} + b_{i+1/2}^{m+\theta}) + \theta l_1 c_1^{m+\theta}, \\
e_{1,0} &= -\theta x_{1/2}(1 - x_{1/2}) 0.5(\overline{a}_{1/2} - b_{i+1/2}^{m+\theta}), \\
e_{1,2} &= -\theta x_{i+1/2}(1 - x_{i+1/2}) b_{i+1/2}^{m+\theta} \phi_{i+1}^{\alpha_i} \Delta_i^{\alpha_i}, \\
F_1 &= u_{h1}^m \left(\frac{l_1}{\Delta t_m} - (1-\theta) \left(x_{i+1/2}(1 - x_{i+1/2}) b_{i+1/2}^{m+(1-\theta)} \phi_i^{\alpha_i} \Delta_i^{\alpha_i} \right.\right. \\
&\left.\left. + x_{1/2}(1 - x_{1/2}) 0.5(\overline{a}_{1/2} + b_{i+1/2}^{m+(1-\theta)}) + l_1 c_1^{m+(1-\theta)} \right) \right) \\
&+ u_{h0}^m (1-\theta) x_{1/2}(1 - x_{1/2}) 0.5(\overline{a}_{1/2} - b_{i+1/2}^{m+(1-\theta)}) \\
&+ u_{h2}^m (1-\theta) x_{i+1/2}(1 - x_{i+1/2}) b_{i+1/2}^{m+(1-\theta)} \phi_{i+1}^{\alpha_i} \Delta_i^{\alpha_i};
\end{aligned}
$$

for $i = 2, \ldots, N-2$

$$
\begin{aligned}
e_{i,i} &= \frac{l_i}{\Delta t_m} + \theta x_{i+1/2}(1 - x_{i+1/2}) b_{i+1/2}^{m+\theta} \phi_i^{\alpha_i} \Delta_i^{\alpha_i} \\
&+ \theta x_{i-1/2}(1 - x_{i-1/2}) b_{i-1/2}^{m+\theta} \phi_i^{\alpha_{i-1}} \Delta_{i-1}^{\alpha_{i-1}} + \theta l_i c_i^{m+(1-\theta)}, \\
e_{i,i-1} &= -\theta x_{i-1/2}(1 - x_{i-1/2}) b_{i-1/2}^{m+\theta} \phi_{i-1}^{\alpha_{i-1}} \Delta_{i-1}^{\alpha_{i-1}}, \\
e_{i,i+1} &= -\theta x_{i+1/2}(1 - x_{i+1/2}) b_{i+1/2}^{m+\theta} \phi_{i+1}^{\alpha_i} \Delta_i^{\alpha_i}, \\
F_i &= u_{hi}^m \left(\frac{l_i}{\Delta t_m} - (1-\theta) \left(x_{i+1/2}(1 - x_{i+1/2}) b_{i+1/2}^{m+(1-\theta)} \phi_i^{\alpha_i} \Delta_i^{\alpha_i} \right.\right. \\
&\left.\left. + x_{i-1/2}(1 - x_{i-1/2}) b_{i-1/2}^{m+(1-\theta)} \phi_i^{\alpha_{i-1}} \Delta_{i-1}^{\alpha_{i-1}} + l_i c_i^{m+(1-\theta)} \right) \right) \\
&+ u_{hi-1}^m (1-\theta) x_{i-1/2}(1 - x_{i-1/2}) b_{i-1/2}^{m+(1-\theta)} \phi_{i-1}^{\alpha_{i-1}} \Delta_{i-1}^{\alpha_{i-1}} \\
&+ u_{hi+1}^m (1-\theta) x_{i+1/2}(1 - x_{i+1/2}) b_{i+1/2}^{m+(1-\theta)} \phi_{i+1}^{\alpha_i} \Delta_i^{\alpha_i}, \ i = 2, \ldots, N-2;
\end{aligned}
$$

for $i = N-1$ if $\alpha_{N-1} > 0$

$$
\begin{aligned}
e_{N-1,N-1} &= \frac{l_{N-1}}{\Delta t_m} + \theta x_{i-1/2}(1-x_{i-1/2}) b_{i-1/2}^{m+\theta} \phi_i^{\alpha_{i-1}} \Delta_{i-1}^{\alpha_{i-1}} + \theta l_{N-1} c_{N-1}^{m+\theta}, \\
e_{N-1,N-2} &= -\theta x_{N-1/2}(1-x_{N-1/2}) b_{N-1/2}^{m+\theta}, \\
e_{N-1,N} &= -\theta x_{i-1/2}(1-x_{i-1/2}) b_{i-1/2}^{m+\theta} \phi_{i-1}^{\alpha_{i-1}} \Delta_{i-1}^{\alpha_{i-1}}, \\
F_{N-1} &= u_{hN-1}^m \left(\frac{l_1}{\Delta t_m} - (1-\theta) \left(x_{i-1/2}(1-x_{i-1/2}) b_{i-1/2}^{m+(1-\theta)} \phi_i^{\alpha_{i-1}} \Delta_{i-1}^{\alpha_{i-1}} \right.\right. \\
&\quad \left.\left. + x_{N-1/2}(1-x_{N-1/2}) b_{N-1/2}^{m+(1-\theta)} + l_{N-1} c_{N-1}^{m+(1-\theta)} \right)\right) \\
&\quad + u_{hN-2}^m (1-\theta) x_{N-1/2}(1-x_{N-1/2}) b_{N-1/2}^{m+(1-\theta)} \\
&\quad + u_{hN}^m (1-\theta) x_{i-1/2}(1-x_{i-1/2}) b_{N-1/2}^{m+(1-\theta)} \phi_{i-1}^{\alpha_{i-1}} \Delta_{i-1}^{\alpha_{i-1}},
\end{aligned}
$$

and if $\alpha_{N-1} \leq 0$ then

$$
\begin{aligned}
e_{N-1,N-1} &= \frac{l_{N-1}}{\Delta t_m} + \theta x_{i-1/2}(1-x_{i-1/2}) b_{i-1/2}^{m+\theta} \phi_i^{\alpha_{i-1}} \Delta_{i-1}^{\alpha_{i-1}} \\
&\quad + \theta x_{N-1/2}(1-x_{N-1/2}) 0.5(\overline{\overline{a}}_{i-1/2}^{m+\theta} - b_{i-1/2}^{m+\theta} + \theta l_{N-1} c_{N-1}^{m+\theta}, \\
e_{N-1,N} &= -\theta x_{N-1/2}(1-x_{N-1/2}) 0.5(\overline{\overline{a}}_{i-1/2}^{m+\theta} + b_{i-1/2}^{m+\theta}, \\
e_{N-1,N-2} &= -\theta x_{i-1/2}(1-x_{i-1/2}) b_{i-1/2}^{m+\theta} \phi_{i-1}^{\alpha_{i-1}} \Delta_{i-1}^{\alpha_{i-1}}, \\
F_{N-1} &= u_{hN-1}^m \left(\frac{l_{N-1}}{\Delta t_m} - (1-\theta) \left(x_{i-1/2}(1-x_{i-1/2}) b_{i-1/2}^{m+(1-\theta)} \phi_i^{\alpha_{i-1}} \Delta_{i-1}^{\alpha_{i-1}} \right.\right. \\
&\quad \left.\left. + x_{N-1/2}(1-x_{N-1/2}) 0.5(\overline{\overline{a}}_{i-1/2}^{m+(1-\theta)} - b_{i-1/2}^{m+(1-\theta)} + l_{N-1} c_{N-1}^{m+(1-\theta)} \right)\right) \\
&\quad + u_{hN-2}^m (1-\theta) x_{i-1/2}(1-x_{i-1/2}) b_{i-1/2}^{m+(1-\theta)} \phi_{i-1}^{\alpha_{i-1}} \Delta_{i-1}^{\alpha_{i-1}} \\
&\quad + u_{hN}^m (1-\theta) x_{N-1/2}(1-x_{N-1/2}) 0.5(\overline{\overline{a}}(x_{N-1/2}) + b(x_{N-1/2}));
\end{aligned}
$$

for $i = N$ if $\alpha_{N-1} > 0$ then

$$
\begin{aligned}
e_{N,N} &= \frac{l_N}{\Delta t_m} + \theta x_{N-1/2}(1-x_{N-1/2}) b_{N-1/2}^{m+\theta} + \theta l_N c_N^{m+\theta},\ e_{N,N-1} = 0, \\
F_N &= u_{hN}^m \left(\frac{l_N}{\Delta t_m} - (1-\theta) \left(x_{N-1/2}(1-x_{N-1/2}) b_{N-1/2}^{m+(1-\theta)} + l_N c_N^{m+\theta} \right)\right),
\end{aligned}
$$

and if $\alpha_{N-1} \le 0$ then

$$
\begin{aligned}
e_{N,N} &= \frac{l_N}{\Delta t_m} + \theta x_{N-1/2}(1 - x_{N-1/2})0.5(\bar{\bar{a}}^{m+\theta}_{N-1/2} + b^{m+\theta}_{N-1/2} + \theta l_N c^{m+\theta}_N, \\
e_{N,N-1} &= -\theta x_{N-1/2}(1 - x_{N-1/2})0.5(\bar{\bar{a}}^{m+\theta}_{N-1/2} - b^{m+\theta}_{N-1/2}), \\
F_N &= u^m_{hN}\left(\frac{l_N}{\Delta t_m} - (1-\theta)\left(x_{N-1/2}(1 - x_{N-1/2})0.5(\bar{\bar{a}}^{m+(1-\theta)}_{N-1/2} + b^{m+(1-\theta)}_{N-1/2})\right.\right. \\
&\quad \left.\left. + l_N c^{m+(1-\theta)}_N\right)\right) \\
&\quad + u^m_{hN-1}(1-\theta)x_{N-1/2}(1 - x_{N-1/2})0.5(\bar{\bar{a}}^{m+(1-\theta)}_{N-1/2} - b^{m+(1-\theta)}_{N-1/2}).
\end{aligned}
$$

Theorem 3. *For any given $m = 1, 2, \ldots, N_t$, if Δt_m is sufficiently small, the system matrix of (18), $\boldsymbol{E}_h$, is an M-matrix.*

Proof. Let us write down the scalar form of (18):

$$
\begin{gathered}
B_0 u^{m+1}_{h0} + C_0 u^{m+1}_{h1} = F_0 \\
A_1 u^{m+1}_{h0} + B_1 u^{m+1}_{h1} + C_1 u^{m+1}_{h2} = F_1 \\
A_2 u^{m+1}_{h1} + B_2 u^{m+1}_{h2} + C_2 u^{m+1}_{h3} = F_2 \\
\dots\dots\dots\dots\dots\dots\dots\dots \\
A_i u^{m+1}_{hi-1} + B_i u^{m+1}_{hi} + C_i u^{m+1}_{hi+1} = F_i \\
\dots\dots\dots\dots\dots\dots\dots\dots \\
A_N u^{m+1}_{hN-1} + B_N u^{m+1}_{hN} = F_N,
\end{gathered}
$$

where

$$
\begin{gathered}
B_0 = \tfrac{h_0}{2\Delta t_m} + \theta e_{0,0},\ C_0 = -\theta e_{0,1}, \\
A_1 = -\theta e_{1,0},\ B_1 = \tfrac{l_1}{\Delta t_m} + \theta e_{1,1},\ C_1 = -\theta e_{1,2}, \\
A_i = -\theta e_{i,i-1},\ B_i = \tfrac{l_i}{\Delta t_m} + \theta e_{i,i},\ C_i = -\theta e_{i,i+1},\ i = 2, 3, \ldots, N-1, \\
A_N = -\theta e_{N,N-1},\ B_N = \tfrac{h_{N-1}}{2\Delta t_m} + \theta e_{N,N}, \\
F_0 = \left(\tfrac{h_0}{2\Delta t_m} - (1-\theta)e_{0,0}\right) u^m_{h0} + (1-\theta)e_{0,1}u^m_{h1}, \\
F_1 = (1-\theta)e_{1,0}u^m_{h0} + \left(\tfrac{l_1}{\Delta t_m} - (1-\theta)e_{1,1}\right) u^m_{h1} + (1-\theta)e_{1,2}u^m_{h2}, \\
F_i = (1-\theta)e_{i,i-1}u^m_{hi-1} + \left(\tfrac{l_i}{\Delta t_m} - (1-\theta)e_{i,i}\right) u^m_{hi} + (1-\theta)e_{i,i+1}u^m_{hi+1}, \\
F_N = (1-\theta)e_{N,N-1}u^m_{hN-1} + \left(\tfrac{h_{N-1}}{2\Delta t_m} - (1-\theta)e_{N,N}\right) u^m_{hN}.
\end{gathered}
$$

Let us first investigate the off-diagonal entries of the system matrix $A_i = -\theta e_{i,i-1}$ and $C_i = -\theta e_{i,i+1}$. From the formulas for $e_{i,j}$ from the above we have

$e_{i,j} > 0$, $i,j = 1,2,\dots,N-1, i \neq j$. That is because

$$b_{i+1/2} \frac{\left(\frac{x_{i+1}}{1-r_{i+1}}\right)^{\alpha_i}}{\left(\frac{x_{i+1}}{1-r_{i+1}}\right)^{\alpha_i} - \left(\frac{x_i}{1-x_i}\right)^{\alpha_i}} = a_{i+1/2}\alpha_i \frac{\left(\frac{x_{i+1}}{1-x_{i+1}}\right)^{\alpha_i}}{\left(\frac{x_{i+1}}{1-x_{i+1}}\right)^{\alpha_i} - \left(\frac{x_i}{1-x_i}\right)^{\alpha_i}}$$
$$= a_{i+1/2}\frac{\alpha_i}{1-\overline{x}_i^{\alpha_i}} > 0,\ 0 < \overline{x}_i = \frac{x_i}{x_{i+1}}\frac{1-x_{i+1}}{1-x_i} < 1.$$

for each $i = 1,2,\dots,N-1$ and each $b_{i+1/2} \neq 0$. We have used that $1-\overline{x}_i^{\alpha_i}$ has just the sign of α_i and this is also true for $b_{i+1/2} \to 0$. Now it is clear that $A_i = -\theta e_{i,i-1}$ and $C_i = -\theta e_{i,i+1}$ are negative.

We should also note that B_i is always positive since Δt_m is small. The situation is different for B_0, C_0, A_1, B_1, C_1 and A_{N-1}, B_{N-1}, C_{N-1}, A_N, B_N. From the first three equations we find

$$u_{h0}^{m+1} = \frac{F_0}{B_0} - \frac{C_0}{B_0}u_{h1}^{m+1},\ u_{h1}^{m+1} = \frac{\triangle_1}{\triangle} - \frac{C_1}{\triangle}u_{h2}^{m+1},$$
$$\triangle = B_1 - \frac{A_1}{B_0}C_0,\ \triangle_1 = F_1 - \frac{A_1}{B_0}F_0,$$
$$\widetilde{B}_2 u_{h2}^{m+1} + C_2 u_{h3}^{m+1} = \widetilde{F}_2,$$
$$\widetilde{B}_2 = B_2 - \frac{A_2C_1}{\triangle},\ \widetilde{F}_2 = F_2 - \frac{\triangle_1}{\triangle}A_2.$$

It is easily to see that when $\triangle > 0$ and $\triangle = O\left(\frac{1}{\Delta t_m}\right)$ then $B_2 = O\left(\frac{1}{\Delta t_m}\right)$ for small Δt_m. Therefore $\widetilde{B}_2 > O\left(\frac{1}{\Delta t_m}\right)$ and $\widetilde{B}_2 > |C_2|$.

In a similar way one can eliminate u_{hN-1}^{m+1} and u_{hN}^{m+1}. As a result we obtain a system of linear algebraic equations with unknowns $u_{h2}^{m+1},\dots,u_{hN-2}^{m+1}$ and coefficients matrix that is an M-matrix.

While $F_3, ..., F_{N-3}$ are non-negative, we have to prove if $\widetilde{F}_2$ and $\widetilde{F}_{N-2}$ are also non-negative. From the formulae for $\widetilde{F}_2$ it follows that when Δt_m is small $\widetilde{F}_2$ is non-negative since $F_2 = O\left(\frac{1}{\Delta t_m}\right)$ and $\triangle, \triangle_1$ are of the same order with respect to Δt_m. $\widetilde{F}_{N-2}$ is being handled the same way as $\widetilde{F}_2$ and also considered non-negative.

Since the load vector $(\widetilde{F}_2, F_3, \dots, F_{N-3}, \widetilde{F}_{N-2})$ is non-negative and the corresponding matrix is an M-matrix we can conclude that $u_{h2}^{m+1},\dots,u_{hN-2}^{m+1}$ are non-negative. Finally, using the formulas for $u_{h0}^{m+1}, u_{h1}^{m+1}, u_{hN-1}^{m+1}, u_{hN}^{m+1}$ one can easily check that they are non-negative too if Δt_m is small. □

Remark 1. Theorem 3 shows that the fully-discretized system (18) satisfies the discrete maximum principle and therefore the above discretization is monotone. This guarantees the following: for non-negative initial function $u_0(x)$ the numerical solution $\mathbf{u}_h^m$, obtained via this method, is also non-negative as expected, because the price of the option is a positive number.

5 Numerical experiments

Numerical experiments, presented in this section, illustrate the properties of the constructed method. We solve numerically various European Test Problems (TP) with different final (initial) conditions and different choices of parameters.

1. $(TP1)$. *Call option* with final condition (2). Parameters: $S_{max} = 700$, $T = 1$, $r = 0.1$, $\sigma = 0.3$, $d = 0.04$ and $E = 400$.
2. $(TP2)$. *Call option with* cash-or-nothing payoff $V(S,T) = H(S - E)$, $S \in (0, \infty)$, where H denotes the Heaviside function. Parameters: $S_{max} = 700$, $T = 1$, $r = 0.1$, $\sigma = 0.3$, $d = 0.04$ and $E = 400$.
3. $(TP3)$. *Call option* with final condition (2). Parameters: $S_{max} = 700$, $T = 1$, $r = 0.1 + 0.02 sin(10Tt)$, $d = 0.06x$ and $E = 400$.
4. $(TP4)$. Portfolio of options. Combinations of different options have step final conditions such as the 'bullish vertical spread' payoff, defined in (3). In this example, we assume that the final condition is a 'butterfly spread' delta function, defined by

$$V(S,T) = \begin{cases} 1, & S \in (S_1, S_2), \\ -1, & S \in (S_2, S_3), \\ 0, & otherwise, \end{cases}$$

   which arises from a portfolio of three types of options with different exercise prices [19].

In the tables below are presented the computed C and L_2 mesh norms of the error $E = u_h - u$ by the formulas

$$\|E\|_C = \max_i \|u_{hi}^{N_t} - u_i^{N_t}\|, \ \|E\|_{L_2} = \sqrt{\sum_{i=0}^{N} l(i) \left(u_{hi}^{N_t} - u_i^{N_t}\right)^2},$$

while the rate of convergence (RC) is calculated using double mesh principle

$$RC = \log_2(E^N / E^{2N}), \quad E^N = \|u_h^N - u^N\|,$$

where $\|.\|$ is the mesh C-norm or L_2-norm, u^N and u_h^N are respectively the exact solution and the numerical solution computed at the mesh with N subintervals.

In Table 1 we show the convergence and the accuracy of the constructed scheme for $\theta = 0.5$, the weight parameter with respect to the time variable, when we numerically solve the model problem with the known exact solution $u(x,t) = \exp(x - t)$ and initial data $u_0(x) = \exp(x)$. We choose this function because it's feature is similar to that of the exact solution to the call option problem. The results, corresponding to problems $TP1$ and $TP3$ with $\Delta t_m = 0.001, m = 0, \ldots, N_t - 1$, are listed in Table 1.

In Table 2 the results are obtained by computations on a *power-graded* mesh for the same values of the parameters and exact solution. This mesh takes into

Table 1.

N	$TP1$ E_∞^N	RC	E_2^N	RC	$TP3$ E_∞^N	RC	E_2^N	RC
80	3.4547e-3	-	2.8008e-4	-	4.8049e-3	-	3.9144e-4	-
160	1.7292e-3	0.998	9.9136e-5	1.498	2.4046-3	0.998	1.3853e-4	1.498
320	8.6503e-4	0.999	3.5068e-5	1.499	1.2028e-3	0.999	4.9000e-5	1.499
640	4.3256e-4	0.999	1.2400e-5	1.499	6.0147e-4	0.999	1.7326e-5	1.499

account *the degeneration at the both ends* of the interval $(0,1)$ and is given by (in the current case $p=2$)

$$x_{i+1} = x_i + \left(i \left(2 \sum_{i=0}^{N/2} i^p \right)^{-1/p} \right)^p, \; i = 1, .., N/2$$

$$x_{i+1} = x_i + \left((N+1-i) \left(2 \sum_{i=0}^{N/2} i^p \right)^{-1/p} \right)^p, \; i = N/2+1, .., N.$$

The time step Δt_m is chosen such that $\Delta t_m = \min_{0 \le i \le N} h(i), m = 0, \ldots, N_t - 1$ with $T = 0.1$.

Table 2.

N	$TP1$ E_∞^N	RC	E_2^N	RC	$TP3$ E_∞^N	RC	E_2^N	RC
20	7.1539e-4	-	3.6478e-4	-	6.2631e-4	-	3.9140e-4	-
40	1.8803e-4	1.927	9.5247e-5	1.947	1.6496e-4	1.924	8.3410e-5	2.230
80	4.8181e-5	1.964	2.4372e-5	1.966	4.2261e-5	1.964	2.1344e-5	1.966
160	1.2195e-5	1.982	6.1671e-6	1.982	1.9695e-5	1.982	5.4011e-6	1.982

In the last table we compute the solutions of the original models $TP2$ and $TP3$. As exact solution we use numerical solution on a very fine grid, i.e. $N = 5120$ with $\Delta t_m = 0.0001, m = 0, \ldots, N_t - 1$. The results are given in Table 3.

The test parameters as well as the obtained results (see also Fig.1-4) are similar to those in [20] as expected.

6 Conclusion

In this article we present a fitted FVM for the generalized Black-Scholes equation (1). The method is applicable to more general Black-Scholes models, for example when $\sigma = \sigma(S,t)$ and $r = r(S,t)$. We may also use any interval $(0,l), \; l > 0$ (here

Table 3.

N	$TP2$				$TP3$			
	E_∞^N	RC	E_2^N	RC	E_∞^N	RC	E_2^N	RC
80	2.9139e-7	-	1.1120e-7	-	2.6809e-3	-	2.1714e-4	-
160	9.9137e-8	1.555	2.8407e-8	1.968	1.3205e-3	1.021	7.4759e-5	1.538
320	5.0468e-8	0.974	7.3863e-9	1.943	6.3931e-4	1.046	2.5443e-5	1.554
640	2.5449e-8	0.987	1.9726e-9	1.904	2.9842e-4	1.099	8.3735e-6	1.603
1280	1.2689e-8	1.003	5.418e-10	1.864	1.2791e-4	1.222	2.5343e-6	1.724

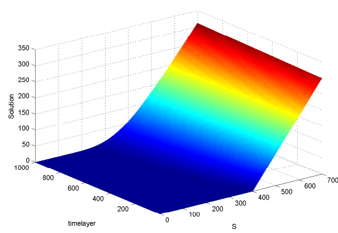

Fig. 1. Test Problem 1

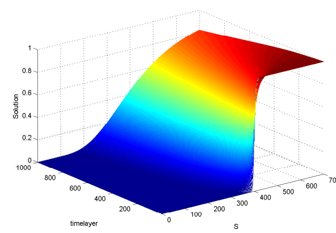

Fig. 2. Test Problem 2

we took $l = 1$ for simplicity) to solve the transformed problem. The main advantage of the developed numerical algorithm is reduction of the computational costs as well as positivity-preserving.

In a forthcoming paper we study the stability and the convergence of the proposed finite volume method.

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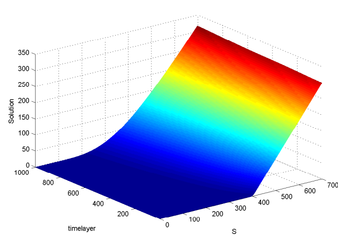

Fig. 3. Test Problem 3

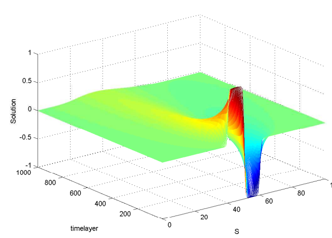

Fig. 4. Test Problem 4

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